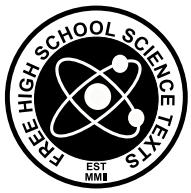

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Contents

Chapter 27

Hyperbolic Functions and Graphs - Grade 11

27.1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of functions.

27.2 Functions of the Form $y = \frac{a}{x+p} + q$

This form of the hyperbolic function is slightly more complex than the form studied in Grade 10.

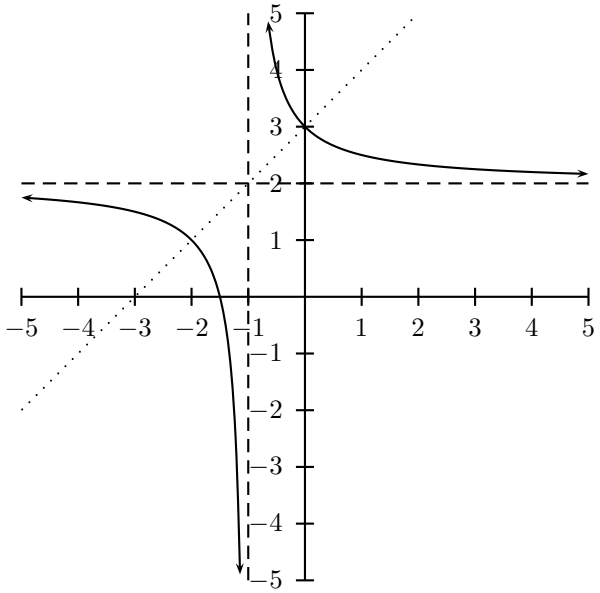


Figure 27.1: General shape and position of the graph of a function of the form $f(x) = \frac{a}{x+p} + q$. The asymptotes are shown as dashed lines.

Activity :: Investigation : Functions of the Form $y = \frac{a}{x+p} + q$

1. On the same set of axes, plot the following graphs:

A $a(x) = \frac{-2}{x+1} + 1$
B $b(x) = \frac{-1}{x+1} + 1$
C $c(x) = \frac{0}{x+1} + 1$
D $d(x) = \frac{+1}{x+1} + 1$
E $e(x) = \frac{+2}{x+1} + 1$

Use your results to deduce the effect of a.

2. On the same set of axes, plot the following graphs:

A $f(x) = \frac{1}{x-2} + 1$
B $g(x) = \frac{1}{x-1} + 1$
C $h(x) = \frac{1}{x+0} + 1$
D $j(x) = \frac{1}{x+1} + 1$
E $k(x) = \frac{1}{x+2} + 1$

Use your results to deduce the effect of p.

3. Following the general method of the above activities, choose your own values of a and p to plot 5 different graphs of $y = \frac{a}{x+p} + q$ to deduce the effect of q.

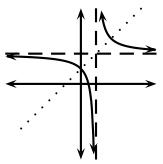
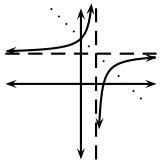
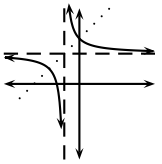
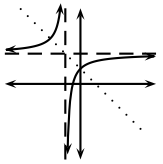
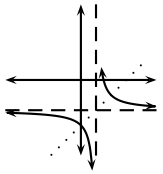
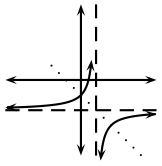
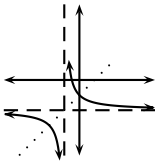
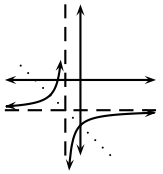
You should have found that the value of a affects whether the graph is located in the first and third quadrants of Cartesian plane.

You should have also found that the value of p affects whether the x-intercept is negative ($p > 0$) or positive ($p < 0$).

You should have also found that the value of q affects whether the graph lies above the x-axis ($q > 0$) or below the x-axis ($q < 0$).

These different properties are summarised in Table 27.1. The axes of symmetry for each graph is shown as a dashed line.

Table 27.1: Table summarising general shapes and positions of functions of the form $y = \frac{a}{x+p} + q$. The axes of symmetry are shown as dashed lines.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q > 0$				
$q < 0$				

27.2.1 Domain and Range

For $y = \frac{a}{x+p} + q$, the function is undefined for $x = -p$. The domain is therefore $\{x : x \in \mathbb{R}, x \neq -p\}$.

We see that $y = \frac{a}{x+p} + q$ can be re-written as:

$$\begin{aligned} y &= \frac{a}{x+p} + q \\ y - q &= \frac{a}{x+p} \\ \text{If } x \neq -p \text{ then: } (y-q)(x+p) &= a \\ x+p &= \frac{a}{y-q} \end{aligned}$$

This shows that the function is undefined at $y = q$. Therefore the range of $f(x) = \frac{a}{x+p} + q$ is $\{f(x) : f(x) \in (-\infty,q) \cup (q,\infty)\}$.

For example, the domain of $g(x) = \frac{2}{x+1} + 2$ is $\{x : x \in \mathbb{R}, x \neq -1\}$ because $g(x)$ is undefined at $x = -1$.

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ (y-2) &= \frac{2}{x+1} \\ (y-2)(x+1) &= 2 \\ (x+1) &= \frac{2}{y-2} \end{aligned}$$

We see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

Exercise: Domain and Range

Determine the range of $y = \frac{1}{x} + 1$.

Given: $f(x) = \frac{8}{x-8} + 4$. Write down the domain of f.

Determine the domain of $y = -\frac{8}{x+1} + 3$

27.2.2 Intercepts

For functions of the form, $y = \frac{a}{x+p} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y-intercept and by setting $y = 0$ for the x-intercept.

The y-intercept is calculated as follows:

$$\begin{aligned} y &= \frac{a}{x+p} + q && (27.1) \\ y_{int} &= \frac{a}{0+p} + q && (27.2) \\ &= \frac{a}{p} + q && (27.3) \end{aligned}$$

For example, the y-intercept of $g(x) = \frac{2}{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ y_{int} &= \frac{2}{0+1} + 2 \\ &= \frac{2}{1} + 2 \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

The x-intercepts are calculated by setting $y = 0$ as follows:

$$y = \frac{a}{x+p} + q \tag{27.4}$$

$$0 = \frac{a}{x_{int}+p} + q \tag{27.5}$$

$$\frac{a}{x_{int}+p} = -q \tag{27.6}$$

$$a = -q(x_{int}+p) \tag{27.7}$$

$$x_{int} + p = \frac{a}{-q} \tag{27.8}$$

$$x_{int} = \frac{a}{-q} - p \tag{27.9}$$

For example, the x-intercept of $g(x) = \frac{2}{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ 0 &= \frac{2}{x_{int}+1} + 2 \\ -2 &= \frac{2}{x_{int}+1} \\ -2(x_{int}+1) &= 2 \\ x_{int}+1 &= \frac{2}{-2} \\ x_{int} &= -1 - 1 \\ x_{int} &= -2 \end{aligned}$$

Exercise: Intercepts

Given:$h(x) = \frac{1}{x+4} - 2$. Determine the coordinates of the intercepts of h with the x- and y-axes.

Determine the x-intercept of the graph of $y = \frac{5}{x} + 2$. Give a reason why there is no y-intercept for this function.

27.2.3 Asymptotes

There are two asymptotes for functions of the form $y = \frac{a}{x+p} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = -p$ and for $y = q$. Therefore the asymptotes are $x = -p$ and $y = q$.

For example, the domain of $g(x) = \frac{2}{x+1} + 2$ is $\{x : x \in \mathbb{R}, x \neq -1\}$ because $g(x)$ is undefined at $x = -1$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$.

From this we deduce that the asymptotes are at $x = -1$ and $y = 2$.

Exercise: Asymptotes

Given:$h(x) = \frac{1}{x+4} - 2$.Determine the equations of the asymptotes of h.

Write down the equation of the vertical asymptote of the graph $y = \frac{1}{x-1}$.

27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$

In order to sketch graphs of functions of the form, $f(x) = \frac{a}{x+p} + q$, we need to calculate determine four characteristics:

1. domain and range
2. asymptotes
3. y-intercept
4. x-intercept

For example, sketch the graph of $g(x) = \frac{2}{x+1} + 2$. Mark the intercepts and asymptotes.

We have determined the domain to be $\{x : x \in \mathbb{R}, x \neq -1\}$ and the range to be $\{g(x) : g(x) \in (-\infty,2) \cup (2,\infty)\}$. Therefore the asymptotes are at $x = -1$ and $y = 2$.

The y-intercept is $y_{int} = 4$ and the x-intercept is $x_{int} = -2$.

Exercise: Graphs

1. Draw the graph of $y = \frac{1}{x} + 2$. Indicate the new horizontal asymptote.
2. Given:$h(x) = \frac{1}{x+4} - 2$. Sketch the graph of h showing clearly the asymptotes and ALL intercepts with the axes.
3. Draw the graph of $y = \frac{1}{x}$ and $y = -\frac{8}{x+1} + 3$ on the same system of axes.
4. Draw the graph of $y = \frac{5}{x-2,5} + 2$. Explain your method.
5. Draw the graph of the function defined by $y = \frac{8}{x-8} + 4$. Indicate the asymptotes and intercepts with the axes.

27.3 End of Chapter Exercises

1. Plot the graph of the hyperbola defined by $y = \frac{2}{x}$ for $-4 \leq x \leq 4$. Suppose the hyperbola is shifted 3 units to the right and 1 unit down. What is the new equation then ?
2. Based on the graph of $y = \frac{1}{x}$, determine the equation of the graph with asymptotes $y = 2$ and $x = 1$ and passing through the point (2; 3).

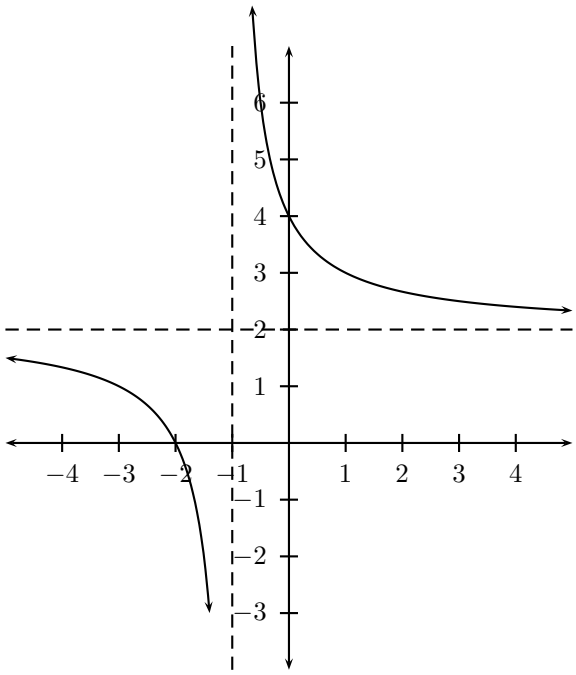


Figure 27.2: Graph of $g(x) = \frac{2}{x+1} + 2$.

Appendix A

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