



FHSST Authors

**The Free High School Science Texts:  
Textbooks for High School Students  
Studying the Sciences  
Mathematics  
Grades 10 - 12**

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## Chapter 27

# Hyperbolic Functions and Graphs - Grade 11

### 27.1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of functions.

### 27.2 Functions of the Form $y = \frac{a}{x+p} + q$

This form of the hyperbolic function is slightly more complex than the form studied in Grade 10.

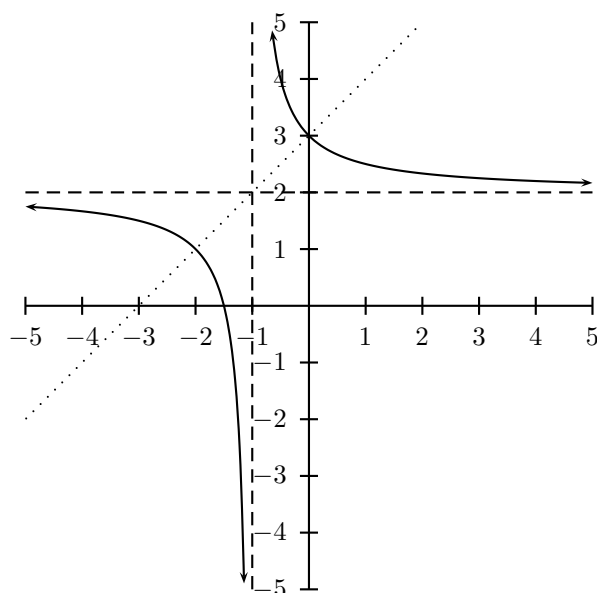


Figure 27.1: General shape and position of the graph of a function of the form  $f(x) = \frac{a}{x+p} + q$ . The asymptotes are shown as dashed lines.

---

#### Activity :: Investigation : Functions of the Form $y = \frac{a}{x+p} + q$

1. On the same set of axes, plot the following graphs:

- A  $a(x) = \frac{-2}{x+1} + 1$   
 B  $b(x) = \frac{-1}{x+1} + 1$   
 C  $c(x) = \frac{0}{x+1} + 1$   
 D  $d(x) = \frac{+1}{x+1} + 1$   
 E  $e(x) = \frac{+2}{x+1} + 1$

Use your results to deduce the effect of  $a$ .

2. On the same set of axes, plot the following graphs:

- A  $f(x) = \frac{1}{x-2} + 1$   
 B  $g(x) = \frac{1}{x-1} + 1$   
 C  $h(x) = \frac{1}{x+0} + 1$   
 D  $j(x) = \frac{1}{x+1} + 1$   
 E  $k(x) = \frac{1}{x+2} + 1$

Use your results to deduce the effect of  $p$ .

3. Following the general method of the above activities, choose your own values of  $a$  and  $p$  to plot 5 different graphs of  $y = \frac{a}{x+p} + q$  to deduce the effect of  $q$ .

You should have found that the value of  $a$  affects whether the graph is located in the first and third quadrants of Cartesian plane.

You should have also found that the value of  $p$  affects whether the  $x$ -intercept is negative ( $p > 0$ ) or positive ( $p < 0$ ).

You should have also found that the value of  $q$  affects whether the graph lies above the  $x$ -axis ( $q > 0$ ) or below the  $x$ -axis ( $q < 0$ ).

These different properties are summarised in Table 27.1. The axes of symmetry for each graph is shown as a dashed line.

Table 27.1: Table summarising general shapes and positions of functions of the form  $y = \frac{a}{x+p} + q$ . The axes of symmetry are shown as dashed lines.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q > 0$				
$q < 0$				

### 27.2.1 Domain and Range

For  $y = \frac{a}{x+p} + q$ , the function is undefined for  $x = -p$ . The domain is therefore  $\{x : x \in \mathbb{R}, x \neq -p\}$ .

We see that  $y = \frac{a}{x+p} + q$  can be re-written as:

$$\begin{aligned} y &= \frac{a}{x+p} + q \\ y - q &= \frac{a}{x+p} \\ \text{If } x \neq -p \text{ then: } (y - q)(x + p) &= a \\ x + p &= \frac{a}{y - q} \end{aligned}$$

This shows that the function is undefined at  $y = q$ . Therefore the range of  $f(x) = \frac{a}{x+p} + q$  is  $\{f(x) : f(x) \in (-\infty, q) \cup (q, \infty)\}$ .

For example, the domain of  $g(x) = \frac{2}{x+1} + 2$  is  $\{x : x \in \mathbb{R}, x \neq -1\}$  because  $g(x)$  is undefined at  $x = -1$ .

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ (y - 2) &= \frac{2}{x+1} \\ (y - 2)(x + 1) &= 2 \\ (x + 1) &= \frac{2}{y - 2} \end{aligned}$$

We see that  $g(x)$  is undefined at  $y = 2$ . Therefore the range is  $\{g(x) : g(x) \in (-\infty, 2) \cup (2, \infty)\}$ .




---

### Exercise: Domain and Range

Determine the range of  $y = \frac{1}{x} + 1$ .

Given:  $f(x) = \frac{8}{x-8} + 4$ . Write down the domain of  $f$ .

Determine the domain of  $y = -\frac{8}{x+1} + 3$

---

## 27.2.2 Intercepts

For functions of the form,  $y = \frac{a}{x+p} + q$ , the intercepts with the  $x$  and  $y$  axis is calculated by setting  $x = 0$  for the  $y$ -intercept and by setting  $y = 0$  for the  $x$ -intercept.

The  $y$ -intercept is calculated as follows:

$$y = \frac{a}{x+p} + q \quad (27.1)$$

$$y_{int} = \frac{a}{0+p} + q \quad (27.2)$$

$$= \frac{a}{p} + q \quad (27.3)$$

For example, the  $y$ -intercept of  $g(x) = \frac{2}{x+1} + 2$  is given by setting  $x = 0$  to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ y_{int} &= \frac{2}{0+1} + 2 \\ &= \frac{2}{1} + 2 \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

The  $x$ -intercepts are calculated by setting  $y = 0$  as follows:

$$y = \frac{a}{x+p} + q \quad (27.4)$$

$$0 = \frac{a}{x_{int} + p} + q \quad (27.5)$$

$$\frac{a}{x_{int} + p} = -q \quad (27.6)$$

$$a = -q(x_{int} + p) \quad (27.7)$$

$$x_{int} + p = \frac{a}{-q} \quad (27.8)$$

$$x_{int} = \frac{a}{-q} - p \quad (27.9)$$

For example, the  $x$ -intercept of  $g(x) = \frac{2}{x+1} + 2$  is given by setting  $x = 0$  to get:

$$\begin{aligned} y &= \frac{2}{x+1} + 2 \\ 0 &= \frac{2}{x_{int} + 1} + 2 \\ -2 &= \frac{2}{x_{int} + 1} \\ -2(x_{int} + 1) &= 2 \\ x_{int} + 1 &= \frac{2}{-2} \\ x_{int} &= -1 - 1 \\ x_{int} &= -2 \end{aligned}$$



### Exercise: Intercepts

Given:  $h(x) = \frac{1}{x+4} - 2$ . Determine the coordinates of the intercepts of  $h$  with the  $x$ - and  $y$ -axes.

Determine the  $x$ -intercept of the graph of  $y = \frac{5}{x} + 2$ . Give a reason why there is no  $y$ -intercept for this function.

### 27.2.3 Asymptotes

There are two asymptotes for functions of the form  $y = \frac{a}{x+p} + q$ . They are determined by examining the domain and range.



We saw that the function was undefined at  $x = -p$  and for  $y = q$ . Therefore the asymptotes are  $x = -p$  and  $y = q$ .

For example, the domain of  $g(x) = \frac{2}{x+1} + 2$  is  $\{x : x \in \mathbb{R}, x \neq -1\}$  because  $g(x)$  is undefined at  $x = -1$ . We also see that  $g(x)$  is undefined at  $y = 2$ . Therefore the range is  $\{g(x) : g(x) \in (-\infty, 2) \cup (2, \infty)\}$ .

From this we deduce that the asymptotes are at  $x = -1$  and  $y = 2$ .




---

**Exercise: Asymptotes**

Given:  $h(x) = \frac{1}{x+4} - 2$ . Determine the equations of the asymptotes of  $h$ .

Write down the equation of the vertical asymptote of the graph  $y = \frac{1}{x-1}$ .

---

### 27.2.4 Sketching Graphs of the Form $f(x) = \frac{a}{x+p} + q$

In order to sketch graphs of functions of the form,  $f(x) = \frac{a}{x+p} + q$ , we need to calculate determine four characteristics:

1. domain and range
2. asymptotes
3.  $y$ -intercept
4.  $x$ -intercept

For example, sketch the graph of  $g(x) = \frac{2}{x+1} + 2$ . Mark the intercepts and asymptotes.

We have determined the domain to be  $\{x : x \in \mathbb{R}, x \neq -1\}$  and the range to be  $\{g(x) : g(x) \in (-\infty, 2) \cup (2, \infty)\}$ . Therefore the asymptotes are at  $x = -1$  and  $y = 2$ .

The  $y$ -intercept is  $y_{int} = 4$  and the  $x$ -intercept is  $x_{int} = -2$ .




---

**Exercise: Graphs**

1. Draw the graph of  $y = \frac{1}{x} + 2$ . Indicate the new horizontal asymptote.
  2. Given:  $h(x) = \frac{1}{x+4} - 2$ . Sketch the graph of  $h$  showing clearly the asymptotes and ALL intercepts with the axes.
  3. Draw the graph of  $y = \frac{1}{x}$  and  $y = -\frac{8}{x+1} + 3$  on the same system of axes.
  4. Draw the graph of  $y = \frac{5}{x-2,5} + 2$ . Explain your method.
  5. Draw the graph of the function defined by  $y = \frac{8}{x-8} + 4$ . Indicate the asymptotes and intercepts with the axes.
- 

## 27.3 End of Chapter Exercises

1. Plot the graph of the hyperbola defined by  $y = \frac{2}{x}$  for  $-4 \leq x \leq 4$ . Suppose the hyperbola is shifted 3 units to the right and 1 unit down. What is the new equation then ?
2. Based on the graph of  $y = \frac{1}{x}$ , determine the equation of the graph with asymptotes  $y = 2$  and  $x = 1$  and passing through the point (2; 3).

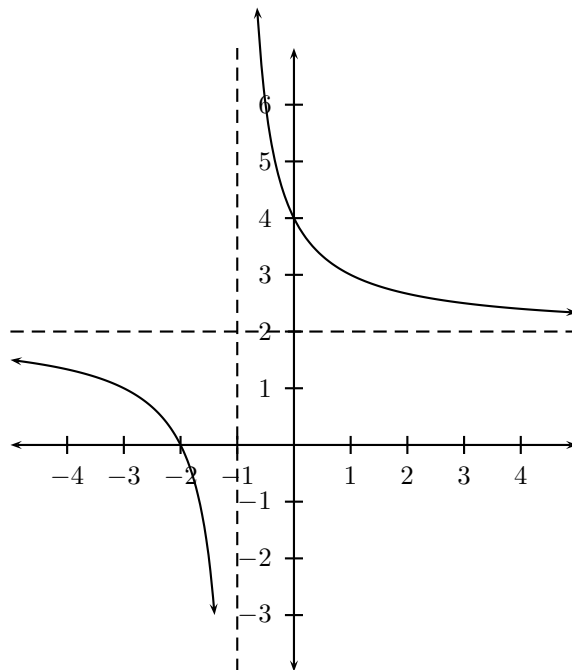


Figure 27.2: Graph of  $g(x) = \frac{2}{x+1} + 2$ .

## Appendix A

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